

THE FINITE-DIMENSIONAL REPRESENTATIONS OF THE RATIONAL CHEREDNIK ALGEBRA OF E_8 WHEN $c = 1/3$

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ABSTRACT. We finish the classification of finite-dimensional irreducible representations of rational Cherednik algebras $H_c(W)$ with equal parameters for exceptional Coxeter groups by resolving the last open case: when $W = E_8$ and the denominator of c is 3.

The finite-dimensional representations in Category $\mathcal{O}_c(W)$ of a rational Cherednik algebra $H_c(W)$ are exactly the cuspidals: $L(\tau) \in \mathcal{O}_c(W)$ is finite-dimensional if and only if $\text{Res}_{\mathcal{O}_c(W')}^{\mathcal{O}_c(W)} L(\tau) = 0$ for all parabolic subgroups $W' \subset W$ [2]. In this sense the classification of the finite-dimensional irreducible representations forms the foundation of the representation theory of $H_c(W)$. When W is an exceptional Coxeter group, work by Chmutova [3], Balagovic-Puranik [1], the author [9],[10], Griffeth-Gusenbauer-Juteau-Lanini [5], and Losev-Shelley-Abrahamson [8] has completed this classification for equal parameters c except when $W = E_8$ and the denominator of c is equal to 3.

For the rest of this note, $c = r/3$, $r \in \mathbb{N}$ and $\gcd(r, 3) = 1$, and we work in Category $\mathcal{O}_c(E_8)$, which contains all finite-dimensional representations. Let $M(\tau) \in \mathcal{O}_c(E_8)$ be the Verma module of lowest weight $\tau \in \text{Irr } E_8$, and let $L(\tau)$ be its simple head. Let $V = \mathbb{C}^8$ be the reflection representation of E_8 . According to Section 5.10 of [5], if $L(\tau)$ is finite-dimensional then:

$$\tau \in \{\text{triv}, V, \phi_{28,8}, \phi_{35,2}, \phi_{50,8}, \phi_{160,7}, \phi_{175,12}, \phi_{300,8}, \phi_{840,13}\} \subset \text{Irr } E_8.$$

Switching to the notation of [7] and [6] for $\text{Irr } E_8$, if $L(\tau)$ is finite-dimensional then:

$$\tau \in \{1_x, 8_z, 28_x, 35_x, 50_x, 160_z, 175_x, 300_x, 840_z\} \subset \text{Irr } E_8.$$

The recent paper of Losev and Shelley-Abrahamson [8] counts that there are exactly eight finite-dimensional irreducibles. To finish the classification it suffices to identify for which τ in the list, $L(\tau)$ is infinite-dimensional.

Theorem. *Let $c = 1/3$. Then $L(50_x)$ is infinite-dimensional.*

Proof. When W is a finite Coxeter group, there is a well-known \mathfrak{sl}_2 -triple $(\mathbf{e}, \mathbf{h}, \mathbf{f})$ in $H_c(W)$, and any finite-dimensional $H_c(W)$ -representation is also a finite-dimensional \mathfrak{sl}_2 -representation [4]. The element $\mathbf{h} := \sum_{i=1}^{\dim V} x_i y_i + y_i x_i$, where $\{x_i\}$ is an orthonormal basis for V^* and $\{y_i\}$ is a dual basis for V , commutes with W , acts on the lowest weight τ of any irreducible $L = L(\tau)$ by a scalar $\mathbf{h}_c(\tau)$, and puts a \mathbb{Z} -grading on L [4]. If L is finite-dimensional then the equality $\dim_{\mathbb{C}} L[-i] = \dim_{\mathbb{C}} L[i]$ must hold for any $i \in \mathbb{Z}$, where $L[i]$ denotes the graded piece of L in degree i , by representation theory of \mathfrak{sl}_2 . We will show that $\dim_{\mathbb{C}} L(50_x)[-1] < \dim_{\mathbb{C}} L(50_x)[1]$, which means that $L(50_x)$ cannot be finite-dimensional.

When $c = 1/3$, $\mathbf{h}_c(\tau) = (\dim V)/2 - c \sum_{\text{reflections } s \in E_8} s|_{\tau} = 4 - (1/3)120\tau(s)/\dim \tau$. Table C.6 of [7] gives the values of $\tau(s)$, from which $\mathbf{h}_c(50_x) = -12$. Next, the decomposition matrix of the Hecke algebra at a root of unity for $e = 3$ and $W = E_8$

given in Table 7.15 of [6] is a submatrix of the decomposition matrix for $\mathcal{O}_{1/3}(E_8)$, but one must tensor the labels τ for rows and columns by the sign representation. Write $\tau' := \tau \otimes \text{sign}$. Look at the column for 50_x in Table 7.15 of [6], which corresponds to the column for $50'_x$ in $\mathcal{O}_{1/3}(E_8)$, and observe that every entry in the column 50_x of Table 7.15 is zero (excepting row 50_x itself of course) until the row labeled 700_{xx} , where there is a 1. Applying Lemma 3.6 from [9], this means that $\dim \text{Hom}(M(50'_x), M(\tau')) = 0$ for all $\tau \in \text{Irr } E_8$, $\mathbf{h}_c(50_x) < \mathbf{h}_c(\tau) < \mathbf{h}_c(700_{xx})$, but $\dim \text{Hom}(M(50'_x), M(700'_{xx})) = 1$. On the other hand, $\mathbf{h}_c(700_{xx}) = 0$ and for all other σ such that $\mathbf{h}_c(\sigma) = 0$, it follows from Table 7.15 of [6] and Lemma 3.6 of [9] that $\dim \text{Hom}(M(50'_x), M(\sigma')) = 0$. Moreover, there is no $\tau \in \text{Irr } E_8$ with $\mathbf{h}_c(\tau) = 1$. By Lemma 3.5 of [9], it follows that for all $700_{xx} \neq \tau \in \text{Irr } E_8$ satisfying $\mathbf{h}_c(50_x) < \mathbf{h}_c(\tau) < 2$, $\dim \text{Hom}(M(\tau), M(50_x)) = 0$, while $\dim \text{Hom}(M(700_{xx}), M(50_x)) = 1$.

The expression in the Grothendieck group of $\mathcal{O}_{1/3}(E_8)$ for $L(50_x)$ in the basis of Verma $M(\tau)$ is therefore: $L(50_x) = M(50_x) - M(700_{xx}) + \sum_{\mathbf{h}_c(\tau) \geq 2} a_{50_x, \tau} M(\tau)$. The graded character of $L(50_x)$ truncated after degree 1 is then $\sum_{k=0}^{13} \binom{7+k}{7} 50t^{-12+k} - \sum_{j=0}^1 \binom{7+j}{7} 700t^j$, and so the graded dimensions of $L(50_x)$ in degrees -1 and 1 (the coefficients of t^{-1} and t in the graded character) are:

$$\begin{aligned} \dim L(50_x)[-1] &= \binom{18}{7} 50 = 1,591,200 \\ \dim L(50_x)[1] &= \binom{20}{7} 50 - \binom{8}{7} 700 = 3,870,400 \end{aligned}$$

Since $\dim L(50_x)[-1] < \dim L(50_x)[1]$, $L(50_x)$ must be infinite-dimensional. \square

It follows that the finite-dimensional irreducible representations of $H_{1/3}(E_8)$ are $L(\tau)$ where $\tau \in \{1_x, 8_z, 28_x, 35_x, 160_z, 175_x, 300_x, 840_z\} \subset \text{Irr } E_8$. By category equivalences, the cases of $c = r/3$ for $r \neq 1$ reduce to the case $c = 1/3$ [4]. This completes the classification.

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